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Relativistic Fluids in Spherically Symmetric Space

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Some of McVittie and Wiltshire's (1977) solutions of Walker's (1935) isotropy conditions for relativistic perfect-fluid spheres are generalized. Solutions are spherically symmetric and conformally flat.

1. INTRODUCTION

For a spherically symmetric metric

$$ds^2 = e^{2\lambda}\, d\eta^2 - e^{2\mu}(d\xi^2 + f^2\, d\theta^2 + f^2 \sin^2\theta\, d\phi^2) \tag{1.1}$$

where f is a function of ξ alone and λ, μ are functions of ξ and η. Einstein's equation for a perfect fluid reduces to

$$e^{2(\mu-\lambda)}(4K_3{}^2 - 2K_1K_4) = K_1K_2 \cdots \tag{1.2}$$

where

$$\begin{aligned}
K_1 &= \mu'' + \lambda'' + \lambda'^2 - \mu'^2 - 2\lambda'\mu' - \frac{f'}{f}(\mu' + \lambda') + \frac{f''}{f^2} - \frac{f'^2}{f^2} + \frac{1}{f^2} \\
K_2 &= \mu'' - \lambda'' + \mu'^2 - \lambda'^2 + \frac{f'}{f}(3\mu' - \lambda') + \frac{f''}{f} + \frac{f'^2}{f^2} - \frac{1}{f^2} \\
K_3 &= \dot{\mu}' - \dot{\lambda}\dot{\mu} \\
K_4 &= \ddot{\mu} - \dot{\lambda}\dot{\mu}
\end{aligned} \tag{1.3}$$

$\mu' \equiv \partial\mu/\partial\xi$, $\dot{\mu} \equiv \partial\mu/\partial\eta$, $\dot{\lambda} \equiv \partial\lambda/\partial\eta$, $\lambda' \equiv \partial\lambda/\partial\xi$, and so on; $(x^1, x^2, x^3, x^4) \equiv (\xi, \theta, \phi, \eta)$ provided the pressure p, density ρ, and velocity v^μ of the fluid are given by

$$8\pi p = G_2{}^2 \qquad 8\pi = e^{-2\mu}K_1 - G_4{}^4 \tag{1.4}$$

$$e^{2\mu}(v^1)^2 = \frac{e^{-2\mu}K_1{}^2}{4e^{-2\lambda}K_3^{2\lambda} - e^{-2\mu}K_1{}^2}$$

$$e^{2\lambda}(v^4)^2 = \frac{4e^{-2\lambda}K_3{}^2}{4e^{-2\lambda}K_3{}^2 - e^{-2\mu}K_1{}^2} \tag{1.5}$$

$$v^2 = 0$$

$$v^3 = 0$$

where $G_\mu{}^\nu = R_\mu{}^\nu - (1/2)\delta_\mu{}^\nu R$, $R_{\mu\nu}$ being the Ricci tensor. (1.2) is the isotropy condition of Walker (1935). This equation can be simplified if we note that there exists a choice of coordinates that keeps the form of (1.1) intact but sets $v^1 = 0$, and hence by (1.4), $K_1 = 0$. (1.2) then merely means $K_3 = 0$. Such coordinates are known as comoving coordinates and such solutions have been obtained by many authors, e.g., Bonnor and Faulkes (1967), Cahill and Taub (1971). However, all such solutions have not been found and obviously the solutions found are the ones that are expressible in terms of simple functions. Since a solution that cannot be expressed in terms of simple functions in a comoving system might be expressible in terms of simple functions in some other coordinate system, say a noncomoving one, McVittie and Wiltshire have felt that there is a need for looking into solutions of (1.1) in a noncomoving system as well, i.e., when

$$K_1 \neq 0$$

and (I)

$$K_3 \neq 0$$

In order to obtain such solutions of (1.2), McVittie and Wiltshire have assumed

$$\lambda = \alpha + \Psi \qquad \mu = \beta + \psi \tag{1.6}$$

where

$$\alpha = \alpha(z) \qquad \beta = \beta(z) \qquad \Psi = \Psi(\eta) \qquad \psi = \psi(\eta) \qquad z = h(\xi) + g(\eta) \tag{1.7}$$

h, g being two functions.

Then they consider two different cases.

Case 1.

$$g = 0 \tag{1.8}$$

Case 2.

$$\psi = ag \qquad a \text{ being a constant} \tag{1.9}$$

In the present work we shall attempt to generalize some of these solutions.

2. SOLUTIONS FOR $g = 0$

In view of (1.7), without loss of generality, one can set

$$Z = \xi \tag{2.1}$$

(1.2) then reduces to

$$e^{2(\beta - \alpha + \psi - \Psi)}[4(\dot{\psi}\alpha')^2 - 2(\ddot{\psi} - \dot{\psi}\dot{\Psi})K_1] - K_1K_2 = 0 \tag{2.2}$$

In view of (1.7) and (2.1), α, β, K_1, K_2 are functions of ξ only. Equation (2.2) has been completely solved by McVittie and Wiltshire subject to further assumptions:

$$\beta = \alpha - \ln f \tag{2.3}$$

and

$$K_1 = 2c\alpha'^2 \tag{2.4}$$

where c is a constant.

Here we attempt to solve (2.2) for the case when (2.3) holds, but (2.4) does not hold, i.e.,

$$K_1 \neq 2c\alpha'^2 \tag{II}$$

for any constant c.

From (1.1), (1.6), (1.7), and (2.1) we note that by a coordinate transformation $\eta' = \int \exp [2(\psi - \psi)]\, d\eta$ one can set

$$\Psi = \psi \tag{2.5}$$

without loss of generality or violation of any of the previous equations or conditions. Since α, β, K_1 are functions of ξ only (2.3) and (II) remain intact:

$$e^{2(\beta - \alpha)}[4(\dot{\psi}\alpha')^2 - 2(\ddot{\psi} - \dot{\psi}^2)K_1] - K_1K_2 = 0 \tag{2.6}$$

Differentiating (2.6) with respect to η,

$$4\alpha'^2 e^{2(\beta - \alpha)} \frac{d}{d\eta}(\dot{\psi}^2) - 2K_1 e^{2(\beta - \alpha)} \frac{d}{d\eta}(\ddot{\psi} - \dot{\psi}^2) = 0 \tag{2.7}$$

Since α, β, K_1 are functions of ξ, and ψ is a function of η, (2.7) can hold only if

$$\dot{\psi}^2 = \text{constant} \tag{2.8}$$

$$\ddot{\psi} - \dot{\psi}^2 = \text{constant} \tag{2.9}$$

However, it is easy to note that (2.9) follows from (2.8) and one can without loss of generality or violation of previous condition set

$$\psi = \eta \tag{2.10}$$

and (2.6) reduces to

$$e^{2(\beta - \alpha)}(4\alpha'^2 + 2K_1) = K_1K_2 \tag{2.11}$$

Putting (1.6), (1.7), (2.1), and (2.3) into (1.1) we note that a coordinate transformation of the form

$$\xi' = \int \frac{d\xi}{f^2}$$

makes

$$\beta = \alpha \qquad f = 1 \tag{2.12}$$

Since such a transformation does not involve η, this will keep (2.5) and (2.10) intact. So do all the other previous equations applicable here. Inequality (II), however, may not hold any longer. But that is immaterial, because once (2.5) and 2.10) remain intact, we do not need (II) any more. From (1.3), (1.6), (1.7), (2.1), (2.11), and (2.12) we get

$$6\alpha'' - 2\alpha'^2 + 3 = 0 \tag{2.13}$$

which can easily be integrated to get α explicitly in terms of ξ. Thus the metric takes the form

$$ds^2 = e^{2(\alpha+\eta)}[d\eta^2 - d\xi^2 - d\Omega^2] \tag{2.14}$$

where $\alpha = \alpha(\xi)$ is given by (2.13).

3. SOLUTIONS FOR $\psi = ag$

In this case McVittie and Wiltshire have further assumed

$$\frac{f''}{f} - \frac{f'^2}{f^2} + \frac{1}{f^2} = 0 \tag{3.1}$$

$$h'' - \frac{f'h'}{f} = bh'^2 \tag{3.2}$$

and

$$h'' = b_1 h'^2 + b_2 \tag{3.3}$$

b, b_1, b_2 being constants.

With these, the isotropy condition reduces to

$$\begin{aligned} 2h'^2\dot{g}^2 e^{2(\beta-\alpha+ag-\psi)}(\beta_{zz} - \alpha_z\beta_z - a\alpha_z)[K_5 - (2a+b)\alpha_z] \\ - K_1\Big\{2e^{2(\beta-\alpha+ag-\psi)}(\beta_z + a)(\ddot{g} - \dot{g}\dot{\psi}) + 2\Big[b_2(2\beta_z - \alpha_z) + \frac{f''}{f}\Big] \\ + h'^2[K_5 + (2b_1 - b)(2\beta_z - \alpha_z)]\Big\} = 0 \end{aligned} \tag{3.4}$$

where K_1 takes the form

$$K_1 = h'^2[\alpha_{zz} + \beta_{zz} + \alpha_z^2 - \beta_z^2 - 2\alpha_z\beta_z + b(\alpha_z + \beta_z)] \tag{3.5}$$

and

$$K_5 = \beta_{zz} - \alpha_{zz} + \beta_z^2 - \alpha_z^2 - b\beta_z \tag{3.6}$$

(3.1), (3.2), and (3.3) have been completely solved by McVittie and Wiltshire to get the following four sets of solutions:

$$\text{Solution (1)}\ f = \xi \qquad h = \frac{b_2\xi^2}{2} + N_2 \qquad b = 0 \qquad b_1 = 0$$

$$\text{Solution (2)}\ f = \xi \qquad h = -\frac{1}{b_1}\ln(N_3\xi) \qquad b = 2b_1 \neq 0 \qquad b_2 = 0$$

$$\text{Solution (3)}\ f = \frac{1}{n_1}\sin(n_1\xi) \qquad h = \frac{1}{b_1}\ln\left[N_3\sin\left(\frac{n_1\xi}{2}\right)\right] \tag{3.7}$$

$$\text{Solution (4)}\ f = \frac{1}{n_1}\sinh(n_1\xi) \qquad h = -\frac{1}{b_1}\ln\left[N_3\sinh\left(\frac{n_1\xi}{2}\right)\right]$$

$$b = 2b_1 \neq 0 \qquad n_1^2 = -4b_1b_2$$

For each of these solutions of (3.1), (3.2), and (3.3), solutions of (3.4) have been obtained by McVittie and Wiltshire under various simplifying assumptions, e.g., taking solution (1), together with assumptions

$$a = 0 \qquad \alpha = \beta \qquad \text{and} \qquad \beta_{zz} - \beta_z^2 \neq 0 \tag{3.8}$$

(3.4) has been completely solved by them. However, in the present work, we shall see that if $\alpha = \beta$ is assumed, then solution (1) is the only solution of (3.1), (3.2), and (3.3) that gives a solution of (3.4). Equation (3.4) will then be solved completely for both $a = 0$ and $a \neq 0$. This is as follows: Using (3.5), (3.6), and (3.8), one can reduce (3.4) to

$$AU + BV + C + Dh'^2 = 0 \tag{3.9}$$

where

$$A = (a + b)\alpha_z(\alpha_{zz} - \alpha_z^2 - a\alpha_z) \qquad B = (\alpha_z + a)(\alpha_{zz} - \alpha_z^2 + b\alpha_z)$$

$$C = (b_z\alpha_z - n_1^2k)(\alpha_{zz} - \alpha_z^2 + b\alpha_z) \qquad D = -\frac{b}{2}(\alpha_{zz} - \alpha_z^2 + b\alpha_z)\alpha_z \tag{3.10}$$

and

$$U = \dot{g}^2e^{2(ag-\Psi)} \qquad V = (\ddot{g} - \dot{g}\dot{\Psi})e^{2(ag-\Psi)}$$

where we note

$$V = \frac{1}{2}\frac{dU}{dg} - aU \tag{3.11}$$

and $k = 0$ for solutions (1) and (2), $k = 1$ for solution (3), $k = -1$ for solution (4), where from (3.7) we have noted that $b = 2b_1$ holds for all four solutions in (3.7) and $f''/f = -n_1{}^2k$. Also, it is to be noted in (3.11) that A, B, C are functions of Z; U and V are functions of η.
where Φ is some function of η.

We shall now prove that $D = 0$. If possible, let $D \neq 0$. Treating η as constant, differentiating (3.9) with respect to h,

$$A_zU + B_zV + C_z + D_zh'^2 + 2Dh'' = 0 \tag{3.12}$$

From (3.3), (3.9), and (3.12)

$$[A(D_z + bD) - A_zD]U + [B(D_z + bD) - B_zD] \\ + [C(D_z + bD) - (C_z + 2b_2D)D] = 0$$

which can be rewritten

$$E(D_z + bD) - E_zD = 0 \tag{3.13}$$

where

$$R = AU + BV + C - 2b_2/bD \tag{3.14}$$

$$E_z \equiv \left.\frac{\partial E}{\partial z}\right|_{\text{treating } \eta \text{ as constant}}$$

and in view of (3.10), $b \neq 0$ followed from $D \neq 0$.

For $D \neq 0$, (3.13) can be integrated with respect to Z; treating η as constant, this gives

$$E = \Phi De^{b_1Z} \tag{3.15}$$

Comparing (3.9) with (3.14) and (3.15) and using (1.6)

$$\Phi e^{bg} = e^{-bh}\left(h'^2 - \frac{2b_2}{b}\right) \tag{3.16}$$

Since the left-hand side of (3.16) is a function of η only and the right-hand side is a function of ξ, they must both be equal to a constant.

However, looking into solutions (1), (2), (3), and (4) of (3.7) we see that none can make the right-hand side of (3.16) a constant. Thus we must have

$$D = 0 \tag{3.17}$$

Also from (3.5), (3.8), and inequality (I)

$$\begin{aligned} \alpha_z &\neq 0 \\ \alpha_{zz} - \alpha_z^2 + b\alpha_z &\neq 0 \end{aligned} \tag{III}$$

From (3.10), (3.17), and inequality (III)

$$b = 0 \tag{3.18}$$

From (3.7), f and h can only be given by solution (1), i.e.,

$$\begin{aligned} f &= \xi \\ h &= (b_2/2)\xi^2 + N_2 \end{aligned} \tag{3.19}$$

and in (3.10),

$$k = 0 \tag{3.20}$$

From (3.5), (3.19), and inequality (I), we must have $b_2 \neq 0$, and thus it is obvious that without loss of generality, one can set

$$b_2 = 2 \qquad N_2 = 0 \tag{3.21}$$

i.e., (3.19) reduces to

$$f = \xi \qquad h = \xi^2 \tag{3.22}$$

From (3.9), (3.17),

$$AU + BV + C = 0 \tag{3.23}$$

where owing to (3.18), (3.20), and (3.21), A, B, C are now given by

$$\begin{aligned} A = a\alpha_z(\alpha_{zz} - \alpha_z^2 - a\alpha_z) \qquad B &= (\alpha_z + a)(\alpha_{zz} - \alpha_z^2) \\ C &= 2\alpha_z(\alpha_{zz} - \alpha_z^2) \end{aligned} \tag{3.24}$$

Equation (3.23) can be completely solved as follows. From (III), (3.18), and (3.24)

$$C \neq 0 \tag{IV}$$

We shall also prove that $B \neq 0$. If possible let $B = 0$. Then from (3.10) and (III),

$$\alpha_z + a = 0$$

Thus

$$\alpha_{zz} = 0$$

These two equations, in view of (3.24), mean

$$A = 0$$

However, $A = 0, B = 0, C \neq 0$ is incompatible with (3.23):

$$B \neq 0 \qquad \text{(V)}$$

From (3.23) and (V)

$$\frac{A}{B} U + V + \frac{C}{B} = 0 \qquad (3.25)$$

Differentiating (3.25) successively with respect to Z, treating η as a constant, and with respect to η, treating Z as constant,

$$\left(\frac{A}{B}\right)_Z \dot{U} = 0$$

Either $U = 0$ or $(A/B)_Z = 0$.

Case 1.

$$U = 0 \qquad \text{i.e., } U = \text{constant} = U_0 \text{ (say)} \qquad (3.26)$$

From (3.11) and (3.26)

$$V = \text{constant} = -U_0$$

(3.23) then reduces to

$$U_0(A - B) + C = 0 \qquad (3.27)$$

where A, B, C are given by (3.24).

Equation (3.27) is an ordinary second differential equation which gives α as a function of Z. (3.26) can be explicitly written

$$\dot{g}^2 e^{2(ag - \Psi)} = U_0 \qquad (3.28)$$

However, from (1.1), (1.6), (1.7), and (1.9) it is obvious that by a suitable transformation of Ψ (where new Ψ is a function of old Ψ only), one can set

$$\Psi = ag \qquad (3.29)$$

From (3.28) we can then set

$$g = \eta(U_0)^{1/2} \qquad U_0 > 0 \qquad (3.30)$$

By (3.22), $Z = h + g$ as in (1.7).

Case 2.

$$\left(\frac{A}{B}\right)_Z = 0 \qquad \text{i.e., } A/B = \text{constant}$$

Since U and V are functions of η, if in (3.25) A/B is a constant, then C/B, being a function of Z, can only be a constant. However, from (3.24) we see that C/B can only be a constant if either $a = 0$ or $\alpha_Z = \text{constant}$.

Subcase 2a.

$$a = 0$$

This is the case that has been worked out by McVittie and Wiltshire. It has been shown that here the metric is the following conformally flat one:

$$ds^2 = e^{2\alpha}[d\eta^2 - d\xi^2 - f^2\, d\theta^2 - f^2 \sin^2\theta\, d\phi^2] \tag{3.31}$$

whereas as before $\alpha = \alpha(Z)$. Z is given by

$$Z = \xi^2 - \eta^2 \tag{3.32}$$

Subcase 2b.

$$\alpha_Z = \text{constant} = \eta_2 \text{ (say)} \tag{3.33}$$

Here A, B, C are constants and in view of (3.11), (3.23) reduces to an ordinary differential equation in U which is readily integrable to give U as a function of η. g can then be taken as an arbitrary function of η, to get Ψ from (3.11). However, to simplify the final expressions we can, as in case 1, set

$$\Psi = ag \tag{3.34}$$

without any loss of generality.

From (3.11), (3.23), (3.24), (3.33), and (3.34), g is given by

$$\ddot{g} + \frac{2n_2}{a_2 + n_2} = 0 \tag{3.35}$$

which has trivial solutions. f and h are given by (3.19), α is given as a function of Z, by (3.33) and $Z = h + g$ as in (1.7).

4. CONCLUSION

Summing up, we see that for a metric of the form (1.1) where λ and μ are given by (1.6) and (1.7), if $g = 0$ and (2.3) is satisfied, then all the solutions of the isotropy condition that are not given by McVittie and Wiltshire are given by a metric of the form (2.14), where $\alpha = \alpha(\xi)$ is given by (2.13).

When λ and μ are given by (1.6), (1.7), but $\psi = ag$, then solutions of (1.2), obtained by solving (3.1), (3.2), (3.3), and (3.4) for $\alpha = \beta$, are as follows.

Solutions reduce to three different cases in all of which the metric has a spherically symmetric and conformally flat form:

$$ds^2 = e^{2(\alpha + ag)}(d\eta^2 - d\xi^2 - \xi^2\, d\theta^2 - \xi^2 \sin^2\theta\, d\phi^2) \tag{3.36}$$

In case 1, α is given as a function of Z by (3.24) and (3.27) and $Z = \xi^2 + \eta(U_0)^{1/2}$ and g is given by (3.30).

An interesting point to note is that in all three cases, the Z = constant curve in the (ξ, η) plane is a conic section, a parabola in case 1, a hyperbola in case 2a, and a hyperbola or ellipse in case 2b. This suggests that although Z was originally introduced to simplify the problem mathematically, Z could have some significance that goes beyond this.

In fact, in case 2a, the case which was already solved by McVittie and Wiltshire, one can check from the expressions for velocity in (1.5) that Z can be interpreted as the velocity potential, i.e., velocity v_μ can be written $v_\mu = \chi Z_{,\mu}$ where χ is some function. Moreover, we can check that p and ρ are functions of Z for case 2a, and hence p and ρ are functionally dependent, i.e., the fluid is barotropic. However, such a simple and interesting interpretation of Z could not be obtained for cases 1 and 2b.

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